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DUALLY EQUIVALENT DECOMPOSITION ALGORITHMS WITH APPLICATION TO SOLVING STAIRCASE STRUCTURES

by

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Abstract

We briefly go over the well known dual relationship between Dantzig-Wolfe Decomposition and Benders Decomposition, in order to develop suitable notation, and then elaborate upon the dual relationship between nested versions of Dantzig-Wolfe and Benders Decomposition. Next we develop a new pair of dually related decompositions termed symmetric Dantzig-Wolfe and symmetric Benders Decomposition. Finally we discuss the advantages and disadvantages of applying nested and symmetric decompositions to structured LP problems, in particular to staircase structures.

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DUALLY EQUIVALENT DECOMPOSITION ALGORITHMS

WITH APPLICATION TO SOLVING STAIRCASE STRUCTURES

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1. Introduction

Solving an LP problem say P by the dual simplex method (Lemke [1]) is equivalent to solving the dual of P by the simplex method (Dantzig [2]). We shall use the term "dually equivalent" to describe a relationship such as this, between two linear programming algorithms.

It is well known that Benders decomposition algorithm [3] is dually equivalent to the Dantzig-Wolfe Decomposition algorithm [4], see e.g., Lasdon [5]. Similarly a nested version of Benders Decomposition algorithm (see e.g., Kallio [6]) is dually equivalent to the nested Dantzig-Wolfe Decomposition algorithm (see Glassey [7], Ho and Manne [8]). This paper is concerned with another pair of dually equivalent algorithms which we believe may a) lead to new and interesting ways to solve staircase systems, b) indicate how different optimization models may be combined. Our paper is organized as follows:

1.1. Overview

In Section 2 we discuss fundamental results. First, we discuss the relationship between Benders and Dantzig-Wolfe Decomposition, primarily to develop notation and lay the basis for the new material.

Nested versions of Dantzig-Wolfe Decomposition have been developed by Glassey [7] and Ho and Manne [8]. By establishing a dual equivalence with nested decomposition, Kallio [6] develops a nested version of Benders algorithm for blocking triangular matrices, but does not go into any detail. Therefore we elaborate a little on this. Next we develop what we call, for want of a better name, symmetric Benders decomposition and symmetric Dantzig-Wolfe decomposition. The symmetric Benders algorithm is related to the tangential approximation method of Geoffrion [9].

Finally, in Section 3, we discuss solution strategies and the advantages and disadvantages of applying nested and symmetric decompositions to structured LP problems, in particular those with staircase structures.

Our aim in this paper is to discuss some basic approaches to solving structured LP problems. Implementational details will be discussed at a later date.

2. Basic Results

2.1.

We consider the primal LP system

$$\left.\begin{aligned} &\text{minimize} \quad \sum_{j=1}^{n} \underline{c}_j^T \underline{x}_j \\ &\text{such that} \quad \sum_{j=1}^{n} A_j \underline{x}_j \geq \underline{b} \\ &\qquad\qquad \underline{x}_j \geq 0 \end{aligned}\right\} \text{(2.1)-P}$$

where

$\underline{c}_j$ and $\underline{x}_j$ are n_j dimensional vectors

A_j is an $(m \times n_j)$ dimensional matrix

and

$\underline{b}$ is an m dimensional vector.

The dual of (2.1)-P is

$$\left.\begin{aligned} &\text{maximize} \quad \underline{\pi}^T \underline{b} \\ &\text{such that} \quad A_j^T \underline{\pi} \leq \underline{c}_j \qquad j = 1, 2, \ldots, n \\ &\qquad\qquad \underline{\pi} \geq 0 \end{aligned}\right\} \text{(2.2)-D}$$

where

$\underline{\pi}$ is an m-dimensional vector

Difficulties of notation often needlessly complicate the description of decomposition algorithms, particularly their nested versions. Therefore we shall confine ourselves in this section to the LP systems (2.1)-P and (2.2)-D with $n = 2$ or $n = 3$, which is quite adequate for our purpose.

A summary of the development of material in this section was given in Section 1.1. Lasdon [p. 382, 5], has described the dual equivalence between Bender's and Dantzig-Wolfe decomposition. In our brief development of nested versions, we shall parallel the development in Lasdon and appeal frequently to his results and notation. Note that nested algorithms have been introduced for specially structured LP's -- staircase structures in Ho and Manne [8], and block triangular structures in Kallio [6]. These are the settings within which the algorithms can be expected to be useful, but they are not restricted in principle to these structures. Since our aim is exposition, we shall let A_j be full matrices. Later, in Section 3, we shall consider special structures.

We shall also employ the notation

$$\left.\begin{aligned} \tilde{A}_j &= (A_1, \ldots, A_j) \\ \tilde{\underline{x}}_j &= (\underline{x}_1, \ldots, \underline{x}_j) \\ \tilde{\underline{c}}_j &= (\underline{c}_1, \ldots, \underline{c}_j) \end{aligned}\right\} \quad (2.3)\text{-N}$$

2.2. Basic Decomposition Algorithms

2.2.1. Benders decomposition algorithm

We first briefly describe Benders algorithm, following Lasdon [5], p. 370]. Consider the problem (2.1)-P with $n = 2$.

$$\left.\begin{aligned} &\text{minimize } \underline{c}_1^T \underline{x}_1 + \underline{c}_2^T \underline{x}_2 \\ &\text{such that } A_1 \underline{x}_1 + A_2 \underline{x}_2 \geq \underline{b} \\ &\qquad \underline{x}_1, \underline{x}_2 \geq 0 \end{aligned}\right\} \text{(2.4)-P2}$$

The method, in essence, fixes $\underline{x}_1$ at some value and solves the resulting LP for $\underline{x}_2$, then adjusts the value of $\underline{x}_1$ and repeats, a procedure known formally as Projection, see Geoffrion [10].

In order to ensure feasibility of the projected LP, $\underline{x}_1$ is restricted to lie in the set (assumed non-empty)

$$R_1 \triangleq \{\underline{x}_1 \geq 0 \mid \exists\, \underline{x}_2 \geq 0 \text{ such that } A_2 \underline{x}_2 \geq \underline{b} - A_1 \underline{x}_1\}$$

Then (2.4)-P2 is equivalent to

$$\min_{\underline{x}_1 \in R_1} \left\{ \underline{c}_1^T \underline{x}_1 + \min_{\underline{x}_2} [\underline{c}_2^T \underline{x}_2 \mid A_2 \underline{x}_2 \geq \underline{b} - A_1 \underline{x}_1 \, , \, \underline{x}_2 \geq 0] \right\} \qquad (2.5)$$

The inner minimization, if dualized, becomes:

BENDERS SUBPROBLEM

$$\left.\begin{aligned} &\text{maximize} \quad (\underline{b} - A_1\, \underline{x}_1)^T\, \underline{u} \\ &\text{such that} \quad A_2^T\, \underline{u} \leq \underline{c}_2 \\ &\qquad \underline{u} \geq 0 \end{aligned}\right\} \text{(2.6)-BSP}$$

and thus (2.5) becomes:

$$\min_{\underline{x}_1 \in R_1} \left\{ \underline{c}_1^T\, \underline{x}_1 + \max_{\underline{u}}[(\underline{b} - A_1\, \underline{x}_1)^T\, \underline{u}\ ,\ \underline{u} \in S] \right\} \tag{2.7}$$

where

$$S \triangleq \{\underline{u} | A_2^T\, \underline{u} \leq \underline{c}_2\ ,\ \underline{u} \geq 0\} \quad .$$

Let us denote the extreme points of S by $\underline{u}_i^p$, $i = 1, \ldots, n_p$ and the extreme rays of S by $\underline{u}_i^r$, $i = 1, \ldots, n_r$. Lasdon [5] establishes two facts

<u>Fact 1</u>: The set R_1 is given by

$$R_1 = \{\underline{x}_1 | (\underline{b} - A_1\, \underline{x}_1)^T\, \underline{u}_i^r \leq 0\ ,\ i = 1, \ldots, n_r\ ,\ \underline{x}_1 \geq 0\}$$

where

$\underline{u}_i^r$ are the extreme rays of the set $C \triangleq \{\underline{u} | A_2^T\, \underline{u} \leq 0\ ,\ \underline{u} \geq 0\}$.

Fact 2: If the inner maximization in (2.7) is unbounded, then the inner minimization in (2.5) would be infeasible, contradicting the assumption that $\underline{x}_1 \in R_1$. Thus in the problem (2.7) we need only consider extreme points of S in the inner maximization.

Thus problem (2.7) can be written as

$$\min_{\underline{x}_1 \in R_1} \left\{ \underline{c}_1^T \underline{x}_1 + \max[(\underline{b} - A_1 x_1)^T \underline{u}_i^p \, , \; i = 1, 2, \ldots, n_p] \right\} \tag{2.8}$$

and this is equivalent to

$$\begin{aligned} &\text{minimize} \quad z \\ &\text{such that} \quad z \geq \underline{c}_1^T \underline{x}_1 + (\underline{b} - A_1 \underline{x}_1)^T \underline{u}_i^p \, , \qquad i = 1, 2, \ldots, n_p \\ &\qquad\qquad\quad 0 \geq (\underline{b} - A_1 \underline{x}_1)^T \underline{u}_i^r \, , \qquad\qquad\quad i = 1, 2, \ldots, n_r \\ &\qquad\qquad\quad \underline{x}_1 \geq 0 \, . \end{aligned}$$

Writing $v \triangleq z - \underline{c}_1^T \underline{x}_1$ this becomes

BENDERS MASTER

$$\left.\begin{aligned} &\text{minimize} \quad c_1^T \underline{x}_1 + v \\ &\text{such that} \quad v \geq (\underline{b} - A_1 \underline{x}_1)^T \underline{u}_i^p \quad i = 1, 2, \ldots, n_p \\ &\qquad\qquad\quad 0 \geq (\underline{b} - A_1 \underline{x}_1)^T \underline{u}_i^r \quad i = 1, 2, \ldots, n_r \\ &\qquad\qquad\quad \underline{x}_1 \geq 0 \end{aligned}\right\} \text{(2.9)-BM}$$

In practice, a relaxation strategy is used, so that only a small fraction of the constraints are employed in a restricted master. Again see Lasdon [5, p. 375], for details.

2.2.2. Dantzig-Wolfe decomposition

We now work with the dual of (2.4)-P2

$$\left.\begin{aligned} &\text{maximize} \quad \underline{b}^T \underline{\pi} \\ &\text{such that} \quad A_1^T \underline{\pi} \leq \underline{c}_1 \\ &\qquad A_2^T \underline{\pi} \leq \underline{c}_2 \\ &\qquad \underline{\pi} \geq 0 \end{aligned}\right\} \text{(2.10)-D2}$$

Treating the constraints $A_2^T \underline{\pi} \leq \underline{c}_2$, $\underline{\pi} \geq 0$ as the subproblem, we have

$$\pi = \sum_{j=1}^{n_p} \lambda_j \underline{u}_j^p + \sum_{j=1}^{n_r} \mu_j \underline{u}_j^r ,$$

with

$$\sum_{j=1}^{n_p} \lambda_j = 1 .$$

The Dantzig-Wolfe [4] master problem becomes

DANTZIG-WOLFE MASTER

$$\left.\begin{aligned}
&\text{maximize} \quad \sum_{j=1}^{n_p} (\underline{b}^T \underline{u}_j^p)\lambda_j + \sum_{j=1}^{n_r} (\underline{b}^T \underline{u}_j^r)\mu_j \\
&\text{such that} \quad \sum_{j=1}^{n_p} (A_1^T \underline{u}_j^p)\lambda_j + \sum_{j=1}^{n_r} (A_1^T \underline{u}_j^r)\mu_j \leq \underline{c}_1 \\
&\qquad \sum_{j=1}^{n_p} \lambda_j = 1 , \quad \lambda_j , \mu_j \geq 0
\end{aligned}\right\} \text{(2.11)-DWM}$$

Extreme points are, of course, developed as needed by a restricted master. Let $\underline{x}_1$ be the dual variables for the first set of constraints, and v the dual variable for the convexity constraint. Then the Dantzig-Wolfe subproblem becomes

$$\left.\begin{aligned}
&\text{maximize} \quad (\underline{b} - A_1 \underline{x}_1)^T \underline{u} \\
&\text{such that} \quad A_2^T \underline{u} \leq \underline{c}_2 \\
&\qquad \underline{u} \geq 0
\end{aligned}\right\} \text{(2.12)-DWSP}$$

2.2.3. Dual equivalence of Benders and Dantzig-Wolfe decomposition

This is easily seen by comparing (2.6)-BSP and (2.12)-DWSP, these being the subproblems corresponding to the two decompositions, and noting that (2.9)-BM is the dual of (2.11)-DWM. It can also be shown that both algorithms employ the same test for optimality.

2.3. Nested Decomposition Algorithms

2.3.1. Nested Bender's decomposition

Let us now consider the system (2.1)-P with $n = 3$. Using the notation (2.3)-N, given at the end of Section 2.1, this can be written in the form

$$\left.\begin{aligned} &\text{minimize} \quad \tilde{\underline{c}}_2^T \tilde{\underline{x}}_2 + \underline{c}_3^T \underline{x}_3 \\ &\text{such that} \quad \tilde{A}_2 \tilde{\underline{x}}_2 + A_3 \underline{x}_3 \geq \underline{b} \\ &\qquad\qquad \tilde{\underline{x}}_2, \underline{x}_3 \geq 0 \end{aligned}\right\} \text{(2.13)-P3}$$

Then following the development of Section 2.2.1 we can apply Benders Decomposition to (2.13)-P3 and we have the subproblem

NESTED BENDER SUBPROBLEM 1

$$\left.\begin{aligned} &\text{maximize} \quad (\underline{b} - \tilde{A}_2 \tilde{\underline{x}}_2)^T \underline{u}_1 \\ &\text{such that} \quad A_3^T \underline{u}_1 \leq \underline{c}_3 \\ &\qquad\qquad \underline{u}_1 \geq 0 \end{aligned}\right\} \text{(2.14)-NBSP1}$$

where a subscript on $\underline{u}$ is introduced to distinguish variables at the first nested level.

Analogously to (2.9)-BM of Section 2.2.1, the first level Benders Master is

NESTED BENDER MASTER 1

$$\left.\begin{aligned} &\text{minimize} \quad \underline{\tilde{c}}_2\, \underline{\tilde{x}}_2 + v_1 \\ &\text{such that} \quad v_1 \geq (\underline{b} - \tilde{A}_2\, \underline{\tilde{x}}_2)^T\, \underline{u}^p_{1i} \quad i = 1, 2, \ldots, n_{1p} \\ &\qquad\qquad\quad 0 \geq (\underline{b} - \tilde{A}_2\, \underline{\tilde{x}}_2)^T\, \underline{u}^r_{1i} \quad i = 1, 2, \ldots, n_{1r} \\ &\qquad\qquad\quad \underline{\tilde{x}}_2 \geq 0 \end{aligned}\right\} \text{(2.15)-NBM1}$$

where $\underline{u}^p_{1i}$, $\underline{u}^r_{1i}$ are defined in an analogous way to the extreme points in (2.9)-BM.

Then if we let

$$U^p_1 = [\underline{u}^p_{11}, \underline{u}^p_{12}, \ldots, \underline{u}^p_{1n_{1p}}]$$

$$U^r_1 = [\underline{u}^r_{11}, \underline{u}^r_{12}, \ldots, \underline{u}^r_{1n_{1r}}]$$

and recalling the definitions of $\tilde{c}_2$, $\tilde{A}_2$ and $\underline{\tilde{x}}_2$ we can write (2.15)-NBM1 as

$$
\begin{array}{llll}
\text{minimize} & \underline{c}_1^T \underline{x}_1 + & \boxed{\begin{array}{l} \underline{c}_2^T \underline{x}_2 + v_1 \\ (U_1^{p^T} A_2)\underline{x}_2 + \underset{\sim}{e}\, v_1 \\ (U_1^{r^T} A_2)\underline{x}_2 \end{array}} & \\
\text{such that} & (U_1^{p^T} A_1)\underline{x}_1 + & & \geq U_1^{p^T} \underline{b} \\
& (U_1^{r^T} A_1)\underline{x}_1 + & & \geq U_1^{r^T} \underline{b} \\
& & \underline{x}_1, \underline{x}_2 \geq 0 &
\end{array}
$$

where $\underline{e}^T = (1, 1, \ldots, 1)$.

If we now apply Benders Decomposition to this system, and let the subproblem correspond to the variables $(\underline{x}_2, v_1)$, we have the second level Benders master given by

NESTED BENDER MASTER 2

$$
\left.
\begin{array}{ll}
\text{minimize} & \underline{c}_1^T \underline{x}_1 + v_2 \\
\text{such that} & v_2 \geq \left[\begin{pmatrix} U_1^{p^T} \underline{b} \\ U_1^{r^T} \underline{b} \end{pmatrix} - \begin{pmatrix} U_1^{p^T} A_1 \\ U_1^{r^T} A_1 \end{pmatrix} \underline{x}_1 \right]^T \underline{u}_{2i}^p \;, \; i=1,2,\ldots,n_{2p} \\
& 0 \geq \left[\begin{pmatrix} U_1^{p^T} \underline{b} \\ U_1^{r^T} \underline{b} \end{pmatrix} - \begin{pmatrix} U_1^{p^T} A_1 \\ U_1^{r^T} A_1 \end{pmatrix} \underline{x}_1 \right]^T \underline{u}_{2i}^r \;, \; i=1,2,\ldots,n_{2r} \\
& \underline{x}_1 \geq 0
\end{array}
\right\} \begin{array}{c} (2.16)- \\ \text{NBM2} \end{array}
$$

and the subproblem is given by

$$\text{maximize} \quad \left[\begin{pmatrix} U_1^{p^T} \underline{b} \\ \\ U_1^{r^T} \underline{b} \end{pmatrix} - \begin{pmatrix} U_1^{p^T} A_1 \\ \\ U_1^{r^T} A_1 \end{pmatrix} \underline{x}_1 \right]^T \underline{u}_2$$

$$\text{such that} \quad \left[\begin{array}{c|c} U_1^{p^T} A_2 & \underline{e} \\ \hline U_1^{r^T} A_2 & 0 \end{array} \right]^T \underline{u}_2 \leq \begin{pmatrix} \underline{c}_2 \\ \hline \underline{1} \end{pmatrix}$$

$$\underline{u}_2 \geq 0$$

and this can be rewritten as

$$\left. \begin{array}{l} \text{maximize} \quad [(\underline{b}^T U_1^p \quad \underline{b}^T U_1^r) - \underline{x}_1^T (A_1^T U_1^p \quad A_1^T U_1^r)]\underline{u}_2 \\ \\ \text{such that} \quad \left[\begin{array}{c|c} A_2^T U_1^p & A_2^T U_1^r \\ \hline \underline{e}^T & \underline{0}^T \end{array} \right] \underline{u}_2 \leq \begin{pmatrix} \underline{c}_2 \\ \hline 1 \end{pmatrix} \\ \\ \qquad \underline{u}_2 \geq 0 \end{array} \right\} \text{(2.17)-NBSP2}$$

2.3.2. Nested Dantzig-Wolfe decomposition

Now consider the dual of (2.13)-P3

$$\left.\begin{aligned} &\text{maximize} \quad \underline{b}^T \underline{\pi} \\ &\text{such that} \quad \tilde{A}_2^T \underline{\pi} \leq \underline{\tilde{c}}_2 \\ &\qquad A_3^T \underline{\pi} \leq \underline{c}_3 \\ &\qquad \underline{\pi} \geq 0 \end{aligned}\right\} \text{(2.18)-D3}$$

Then carry out the standard Dantzig-Wolfe decomposition cf §2.2.2 we have the master given, analogously to (2.11)-DWM by

NESTED DANTZIG-WOLFE MASTER 1

$$\left.\begin{aligned} &\text{maximize} \quad \sum_{i=1}^{n_{1p}} (\underline{b}^T \underline{u}_{1i}^p)\lambda_{1i} + \sum_{i=1}^{n_{1r}} (\underline{b}^T \underline{u}_{1i}^r)\mu_{1i} \\ &\text{such that} \quad \sum_{i=1}^{n_{1p}} (\tilde{A}_2^T \underline{u}_{1i}^p)\lambda_{1i} + \sum_{i=1}^{n_{1r}} (\tilde{A}_2^t \underline{u}_{1i}^r)\mu_{1i} \geq \tilde{c}_2 \\ &\qquad \sum_{i}^{n_{1p}} \lambda_{1i} = 1 \\ &\qquad \lambda_{1i}, \mu_{1i} \geq 0 \end{aligned}\right\} \text{(2.19)-NDWM1}$$

where $\underline{u}^p_{1i}$, $i = 1, \ldots, n_{1p}$ and $\underline{u}^r_{1i}$, $i = 1, \ldots, n_{1r}$ are the extreme points and extreme rays of the subproblem which is given by

NESTED DANTZIG-WOLFE SUBPROBLEM 1

$$\text{maximize} \quad (\underline{b} - \tilde{A}_2 \underline{\bar{x}}_2)^T \underline{\pi}$$

$$\text{such that} \quad A_3^T \underline{\pi} \leq \underline{c}_3 \qquad \text{(2.20)-NDWSP1}$$

$$\underline{\pi} \geq 0$$

where $\underline{\bar{x}}_2$ are the dual variables of rows other than the convexity row in (2.19). If we substitute for $\tilde{A}_2$ and $\tilde{c}_2$ in (2.19)-NDWM1 we get

$$\text{maximize} \quad \sum_i (\underline{b}^T \underline{u}^p_{1i})\lambda_{1i} + \sum_i (\underline{b}^T \underline{u}^r_{1i})\mu_{1i}$$

$$\text{such that} \quad \sum_i (A_1^T \underline{u}^p_{1i})\lambda_{1i} + \sum_i (A_1^T \underline{u}^r_{1i})\mu_{1i} \leq \underline{c}_2$$

$$\sum_i (A_2^T \underline{u}^p_{1i})\lambda_{1i} + \sum_i (A_2^T \underline{u}^r_{1i})\mu_{1i} \leq \underline{c}_2$$

$$\sum_i \lambda_{1i} = 1$$

$$\lambda_{1i}, \mu_{1i} \geq 0 .$$

Writing

$$U_1^p = (\underline{u}_{11}^p, \ldots, \underline{u}_{1n_{1p}}^p)$$

$$U_1^r = (\underline{u}_{11}^r, \ldots, \underline{u}_{1n_{1r}}^r)$$

and

$$\underline{\lambda}_1^T = (\lambda_{11}, \ldots, \lambda_{1n_{1p}})^T$$

and

$$\underline{\mu}^T = (\mu_{11}, \ldots, \mu_{1n_{1r}})^T$$

this becomes

NESTED DANTZIG-WOLFE MASTER 1

$$\left.\begin{array}{rl} \text{maximize} & (\underline{b}^T U_1^p)\underline{\lambda}_1 + (\underline{b}^T U_1^r)\underline{\mu}_1 \\ \text{such that} & (A_1^T U_1^p)\underline{\lambda}_1 + (A_1^T U_1^r)\underline{\mu}_1 \leq \underline{c}_1 \\ & \boxed{\begin{array}{ll} (A_2^T U_1^p)\underline{\lambda}_1 + (A_2^T U_1^r)\underline{\mu}_1 \leq \underline{c}_2 & \\ \quad \underline{e}^T \underline{\lambda}_1 & = 1 \\ \underline{\lambda}_1, \underline{\mu}_1 \geq 0 & \end{array}} \end{array}\right\} \text{(2.21)-NDWM1}$$

Now if we repeat the Dantzig-Wolfe decomposition, letting the subproblem correspond to the boxed set of constraints, we obtain the second level master as

NESTED DANTZIG-WOLFE MASTER 2

$$\left.\begin{aligned}
&\text{maximize} \sum_{i=1}^{n_{2p}} [(\underline{b}^T U_1^p \mid \underline{b}^T U_1^r)\underline{u}_{2i}^p]\lambda_{2i} + \sum_{i=1}^{n_{2r}} [(\underline{b}^T U_1^p \mid \underline{b}^T U_1^r)\underline{u}_{2i}^r]\mu_{2i} \\
&\text{such that} \sum_{i=1}^{n_{2p}} [(A_1^T U_1^p \mid A_1^T U_1^r)\underline{u}_{2i}^p]\lambda_{2i} + \sum_{i=1}^{n_{2r}} [(A_1^T U_1^p \mid A_1^T U_1^r)\underline{u}_{2i}^r]\mu_{2i} \le c_1 \\
&\sum_{i=1}^{n_{2p}} \lambda_{2i} = 1 \\
&\lambda_{2i}, \mu_{2i} \ge 0 \ .
\end{aligned}\right\} \text{(2.22)-NDWM2}$$

If we denote the dual variables corresponding to the first set of constraints, by $\underline{x}_1^T$, our subproblem is then given by

$$\left.\begin{aligned}
&\text{maximize} \quad [(\underline{b}^T U_1^p \mid \underline{b}^T U_1^r) - \underline{x}_1^T(A_1^T U_1^p \mid A_1^T U_1^r)]\underline{u}_2 \\
&\text{such that} \quad \left[\begin{array}{c|c} A_2^T U_1^p & A_2^T U_1^r \\ \hline \underline{e}^T & \underline{0}^T \end{array}\right] \underline{u}_2 \le \left(\begin{array}{c} \underline{c}_2 \\ \hline 1 \end{array}\right)
\end{aligned}\right\} \text{(2.23)-NDWSP2}$$

2.3.3. Dual equivalence of nested Benders decomposition and nested Dantzig-Wolfe decomposition

If we denote the dual variables of (2.19)-NDWM1 by $\underset{\sim}{x}_2$ and v_1 then we can see that (2.15)-NBM1 is the dual of (2.19)-NDWM1. Also (2.14)-NBSP1 is the same as (2.20)-NDWSP1. Again, if we compare (2.16)-NBM2 with (2.22)-NDWM2 and let $\underline{x}_1$ and v_2 be the dual variables of (2.22)-NDWM2, we see that (2.22)-NDWM2 and (2.16)-NBM2 are duals. Also (2.17)-NBSP2 is the same as (2.23)-NDWSP2. Note again that we will deal in practice with restricted master problems.

2.4. Symmetric Decomposition Algorithms

2.4.1. Symmetric Benders decomposition

Let us return to problem (2.4)-P2 of Section 1.1

$$\left.\begin{aligned} &\text{minimize } \underline{c}_1^T \underline{x}_1 + \underline{c}_2^T \underline{x}_2 \\ &\text{such that } A_1 \underline{x}_1 + A_2 \underline{x}_2 \geq \underline{b} \\ &\underline{x}_1, \underline{x}_2 \geq 0 \end{aligned}\right\} \text{(2.24)-P2}$$

We shall employ the following notation of Geoffrion [9]

$$Y_i \triangleq \{\underline{y}_i \in \mathbb{R}^m \mid \exists\, \underline{x}_i \text{ such that } A_i \underline{x}_i \geq \underline{y}_i, \quad \underline{x}_i \geq 0\}$$

for $i = 1, 2$.

$$v_i(y_i) \triangleq \min_{x_i \geq 0} \{\underline{c}_i^T \underline{x}_i | A_i \underline{x}_i \geq \underline{y}_i\} \quad .$$

If the set is infeasible for the given y_i, then $v_i(\underline{y}_i) \triangleq +\infty$.

Geoffrion [9] shows that (2.24)-P2 is equivalent to

$$\left.\begin{aligned} &\text{minimize} \quad v_1(\underline{y}_1) + v_2(\underline{y}_2) \\ &\text{such that} \quad \underline{y}_1 + \underline{y}_2 \geq \underline{b} \\ &\underline{y}_i \in Y_i \,, \qquad i = 1, 2 \quad . \end{aligned}\right\} \text{(2.25)-P2}$$

Fact 3: In an essentially identical proof to that used by Lasdon [3] to establish Fact 1, Section 2.2.1, we have

$$\{\underline{y}_i \in Y_i \leftrightarrow \underline{y}_i^T \underline{u}_{ij}^r \leq 0 \ , \ j = 1, 2, \ldots, n_{ir}\}$$

for $i = 1, 2$, where $\underline{u}_{ij}^r$, $j = 1, 2, \ldots, n_{ir}$ are the extreme rays of the sets $C_i \triangleq \{\underline{u} | A_i^T \underline{u} \leq 0, \underline{u} \geq 0\}$, $i = 1, 2$. □

Consider now the objective function of (2.25)-P2

$$v_1(\underline{y}_1) + v_2(\underline{y}_2) = \min_{\underline{x}_1 \geq 0} \{\underline{c}_1^T \underline{x}_1 | A_1 \underline{x}_1 \geq \underline{y}_1\} + \min_{x_2 \geq 0} \{\underline{c}_2^T \underline{x}_2 | A_2 \underline{x}_2 \geq \underline{y}_2\}$$

If we dualize each of these minimizations, we can write this as

$$v_1(\underline{y}_1) + v_2(\underline{y}_2) = \max_{\underline{u}_1 \geq 0} \{\underline{y}_1^T \underline{u}_1 | A_1^T \underline{u}_1 \leq \underline{c}_1\} + \max_{\underline{u}_2 \geq 0} \{\underline{y}_2^T \underline{u}_2 | A_2^T \underline{u}_2 \leq \underline{c}_2\}$$

Fact 4: Again, analogously to Fact 2 of Section 2.2.1, if either maximization problem above went to $+\infty$ along an extreme ray, the corresponding dual minimization problem *would be infeasible.* This would contradict the constraint $y_i \in Y_i$. Therefore we need only consider extreme points of the feasible sets

$$\{\underline{u}_i | A_i^T \underline{u}_i \leq \underline{c}_i \, , \, \underline{u}_i \geq 0\} \, , \qquad i = 1, 2 \; .$$

Let us denote these extreme points by $\underline{u}_{ij}^p$, $j = 1, 2, \ldots, n_{ip}$ for $i = 1, 2$.

Then (2.25)-P2 can be written as

$$\min_{\underline{y}_1, \underline{y}_2} \; [\max\{\underline{y}_1^T \underline{u}_{1j}^p \, , \, j = 1, \ldots, n_{1p}\} + \max\{\underline{y}_2^T \underline{u}_{2j}^p \, , \, j = 1, 2, \ldots, n_{2p}\}]$$

$$\text{such that } \; \underline{y}_1 + \underline{y}_2 \geq \underline{b}$$

$$\underline{y}_i \in Y_i \, , \qquad i = 1, 2$$

Thus, finally, this can be written as the symmetric Bender Master problem

SYMMETRIC BENDER MASTER

$$
\left.
\begin{aligned}
&\text{minimize} \quad z_1 + z_2 \\
&\text{such that} \quad z_1 \geq \underline{y}_1^T \, \underline{u}_{1j}^p \, , && j = 1, \ldots, n_{1p} \\
&\qquad\qquad\quad 0 \geq \underline{y}_1^T \, \underline{u}_{1j}^r \, , && j = 1, \ldots, n_{1r} \\
&\qquad\qquad\quad z_2 \geq \underline{y}_2^T \, \underline{u}_{2j}^p \, , && j = 1, \ldots, n_{2p} \\
&\qquad\qquad\quad 0 \geq \underline{y}_2^T \, \underline{u}_{2j}^r \, , && j = 1, \ldots, n_{2r} \\
&\qquad\qquad\quad \underline{b} \leq \underline{y}_1 + \underline{y}_2 \, .
\end{aligned}
\right\} \text{(2.26)-SBM}
$$

and the symmetric Bender subproblems are given by

SYMMETRIC BENDER SUBPROBLEM i

$$
\left.
\begin{aligned}
&\text{maximize} \quad \underline{y}_i^T \, \underline{u}_i \\
&\text{such that} \quad A_i^T \, \underline{u}_i \leq \underline{c}_i \\
&\qquad\qquad\qquad \underline{u}_i \geq 0
\end{aligned}
\right\} \text{(2.27)-SBSPi}
$$

for $i = 1, 2$.

Note that the master has block angular form, and that again, in practice, we shall deal with a restricted master problem.

2.4.2. Symmetric Dantzig-Wolfe decomposition

Consider again the problem (2.10)-D2

$$\left.\begin{aligned} &\text{maximize} \quad \underline{b}^T \underline{\pi} \\ &\text{such that} \quad A_1^T \underline{\pi} \leq \underline{c}_1 \\ &\qquad\qquad\quad A_2^T \underline{\pi} \leq \underline{c}_2 \\ &\qquad\qquad\quad \underline{\pi} \geq 0 \end{aligned}\right\} \quad (2.28)\text{-D2}$$

A feasible solution $\underline{\pi}_1$ of the set $S_1 \triangleq \{\underline{\pi} \mid A_1^T \underline{\pi} \leq \underline{c}_1, \ \underline{\pi} \geq 0\}$ satisfies

$$\left.\begin{aligned} &\underline{\pi}_1 = \sum_{j=1}^{n_{1p}} \lambda_{1j} \underline{u}_{1j}^p + \sum_{j=1}^{n_{1r}} \mu_{1j} \underline{u}_{1j}^r \\ &\text{with} \\ &\sum_{j=1}^{n_{1p}} \lambda_{1j} = 1 \\ &\lambda_{1j} \geq 0, \qquad \mu_{1j} \geq 0 \end{aligned}\right\}$$

where u_{1j}^{p}, $j = 1, \ldots, n_{1p}$ are the extreme points of S_1, and u_{1j}^{r}, $j = 1, \ldots, n_{1r}$ are the extreme rays of S_1.

Likewise a feasible solution $\underline{\pi}_2$ of $S_2 = \{\underline{\pi}_2 | A_2^T \underline{\pi} \leq c_2, \underline{\pi} \geq 0\}$ satisfies

$$\underline{\pi}_2 = \sum_{j=1}^{n_{2p}} \lambda_{2j} \underline{u}_{2j}^{p} + \sum_{j=1}^{n_{2r}} \mu_{2j} \underline{u}_{2j}^{r}$$

$$\sum_{j=1}^{n_{2p}} \lambda_{2j} = 1$$

$$\lambda_{2j}, \mu_{2j} \geq 0$$

Then (2.28)-D2 can be written as

SYMMETRIC DANTZIG-WOLFE DECOMPOSITION MASTER

$$\left.\begin{array}{lllr}
\text{maximize} & \underline{b}^T \underline{\pi} & & \\
\text{such that} & \underline{\pi} - \left(\sum_j \lambda_{1j} \underline{u}_{1j}^{p} + \sum_j \mu_{1j} \underline{u}_{1j}^{r}\right) & & = 0 \\
 & \underline{\pi} & - \left(\sum_j \lambda_{2j} \underline{u}_{2j}^{p} + \sum_j \mu_{2j} \underline{u}_{2j}^{r}\right) & = 0 \\
 & \quad \sum_j \lambda_{1j} & & = 1 \\
 & & \sum_j \lambda_{2j} & = 1 \\
 & \multicolumn{2}{c}{\underline{\pi}, \lambda_{1j}, \lambda_{2j}, \mu_{1j}, \mu_{2j} \geq 0 .} &
\end{array}\right\} \text{(2.29)-SDWM}$$

If we denote the dual variables corresponding to the first two sets of constraints by $\underline{y}_1$ and $\underline{y}_2$, then the two subproblems for symmetric Dantzig-Wolfe Decomposition are given by

SYMMETRIC DANTZIG-WOLFE SUBPROBLEM i

$$\text{maximize} \quad \underline{y}_i^T \underline{u}_i$$

$$\text{such that} \quad A_i^T \underline{u}_i \leq \underline{c}_i \qquad \text{(2.30)-SDWSPi}$$

$$\underline{u}_i \geq 0$$

2.4.3. Dual equivalence of symmetric Benders and symmetric Dantzig-Wolfe decomposition

This is immediately seen by noting that (2.27)-SBSPi and (2.30)-SDWSPi are identical, and that (2.26)-SBM is dual to (2.29)-SDWM. Again, in practice we use restricted master problems.

3. Concluding Remarks on Solution Strategies and Application to Special Structures

In Section 2 we dealt with algorithms for solving (2.1)-P and (2.2)-D, confining the discussion to the case $n = 2$ or 3. It is easy to see how the algorithms extend to arbitrary n. In this section we elaborate upon these algorithms discussing, in particular, their advantages and disadvantages. We shall be primarily concerned with their application to staircase structures. Figure 3.1a) and 3.1b) illustrate two dual staircase LP's, which are special cases of general LP's illustrated in Figures 3.2a) and 3.2b)

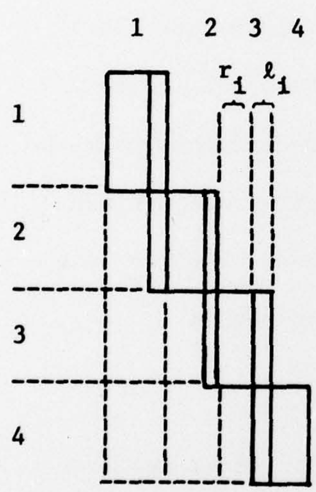

Figure 3.1a)

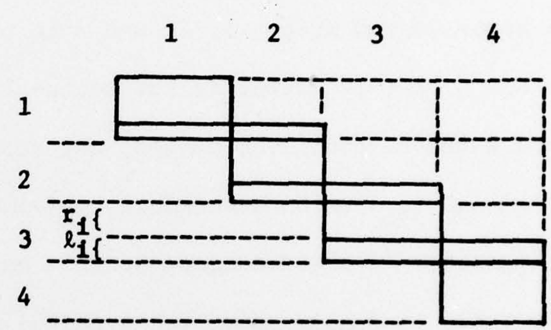

Figure 3.1b)

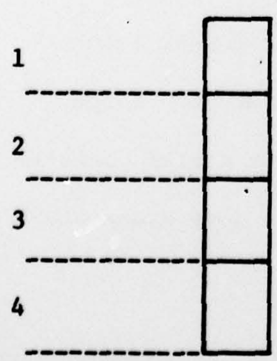

Figure 3.2a)

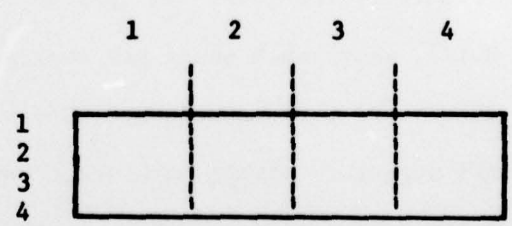

Figure 3.2b)

Note that in staircase structures there are often only a few linking columns in Figure 3.1a) and correspondingly only a few linking rows in the LP dual to it, Figure 3.1b).

3.1. Nested Decomposition Algorithms

The algorithms for Glassey [7] and Ho and Manne [8] were developed for staircase LP's of the form shown in Figure 3.1b). The algorithm of Kallio [6] is a generalization of these to the case when the matrix is block upper triangular.

In these algorithms, stage 4 would correspond to the first subproblem and stages 1, 2, and 3 to the first level master. Then stage 3 of this master is the second level subproblem and stages 1 and 2 the second level master, and so on. A major contribution of Ho [11] was to develop a workable implementation of the procedure and demonstrate its effectiveness, since many details must be worked out in order to specify a workable nested decomposition method.

We should note the following:

a) The nested decomposition method of Section 2.3.2 works best for structures of the form 3.1b), since each master will then have relatively few rows. If applied to the staircase structure 3.1a), where each stage has many more rows than columns, each master and thus subsequent subproblems would become very large.

b) All structure <u>within</u> each stage can be lost in the nested Dantzig-Wolfe Decomposition. Thus, if the constraints of some stage are

implicitly defined, e.g., by a network, we cannot exploit this added structure in the problem.

c) The procedure lacks symmetry with respect to interchange of stages.

d) There is an inherent propogation and build up of error in the nested decomposition process. Errors in the extreme points of a subproblem, result in errors in the columns of the master to which these extreme points are transmitted, which in turn result in errors in the next level of the decomposition. For a more detailed discussion see Nazareth [1]2.

e) A post-optimality procedure (called Phase 3) is needed to recover the solution (see Ho [11]).

f) Finally, note that the nested Dantzig-Wolfe decomposition can be carried out in other ways. Thus in Figure 3.1b) stages 3 and 4 could be the first level subproblem and stages 1 and 2 the first level master then each of these could be decomposed in turn into a master and subproblem. A serious disadvantage of such a procedure is that a Phase 3 process would be model at intermediate stages in order to recover an extreme point of a subproblem, for transmission to a master.

By the dual equivalence discussed in Section 2.3.3, we would expect the Nested Benders algorithm to work best on structures of the form 3.1a) and 3.2a). Because we wish to have primal feasibility explicitly in 3.2b) we might consider applying nested Benders decomposition to this structure. However, note that we would then have the

disadvantages of nested Dantzig-Wolfe decomposition applied to 3.2a) i.e., at first sight it appears as though we get master problems with a large number of rows, though there may be ways to get around this by working with the dual.

3.2. Symmetric Decomposition Algorithms

In contrast with the Nested Dantzig-Wolfe decomposition algorithm, the symmetric Dantzig-Wolfe algorithm applies most naturally to structures of the form 3.1a) and 3.2a), since the size of the symmetric Dantzig-Wolfe master is determined by the number of columns. Let ℓ_i denote the number of linking columns for stage i in Figure 3.1a) and r_i the number of remaining columns for stage i (correspondingly ℓ_i and r_i are the number linking rows and remaining rows in Figure 3.1b)). If we decompose into 4 subproblems, one for each stage in Figure 3.1a), the symmetric Dantzig-Wolfe master has $\sum_{i=1}^{4} r_i + 2\sum_{i=1}^{3} \ell_i + 4$ rows and a structure illustrated in Figure 3.3. Note that the shaded blocks correspond to dense matrices, a potential disadvantage.

The first portion of the matrix has $\sum_1^4 r_i + \sum_1^3 \ell_i$ columns. In general, if extreme points of subproblems are not degenerate, and cancellation does not occur, we would expect that variables corresponding to this first part to have non-zero values. Thus we would expect a basic solution of the symmetric master to have approximately $\sum_1^3 \ell_i + 4$ columns from the dense block diagonal part of the matrix. If ℓ_i is small, these would be relatively few. Operations involving

a basis, for example, LU factorization, in this system can then be handled more efficiently than corresponding operations in the sparse staircase system 3.1b).

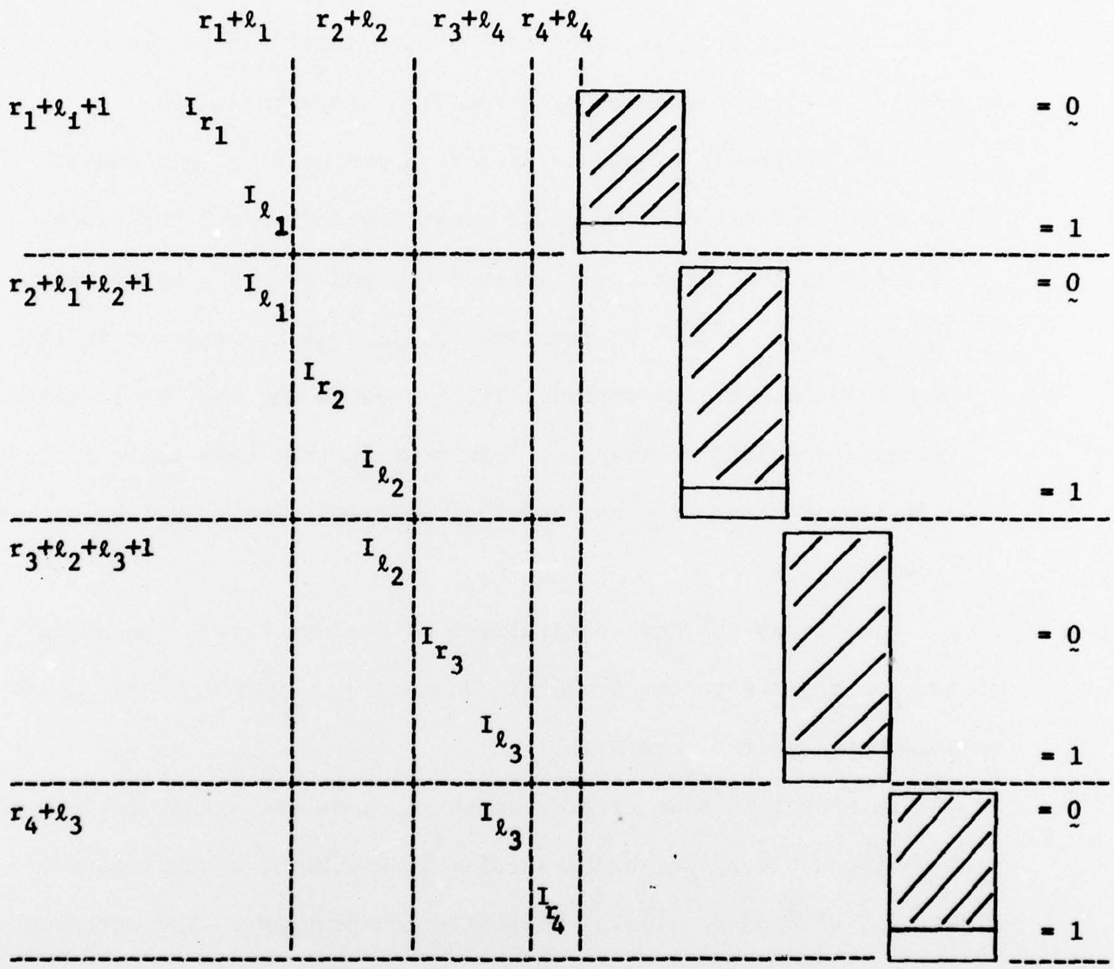

Figure 3.3

Notation for Figure 3.3: I_{ℓ_i} **denotes an** $\ell_i \times \ell_i$ **identity matrix.**

In contrast to the nested algorithms note:

a) No Phase 3 is required to recover the solution.

b) Special structure within a subproblem is not lost. In particular different models can be integrated into a single model and at the same time the special structure of each model can be exploited.

c) There is no transmission of error from stage to stage.

d) It is also possible to do symmetric decomposition in a nested manner. For example, we could treat stages 1 and 2 and stages 3 and 4 as the first level subproblems and obtain a master with $\sum_1^3 \ell_i + \sum_1^4 r_i + \ell_2 + 2$ rows and $\sum_1^3 \ell_i + \sum_1^4 r_i$ columns in the front portion of the matrix. The procedure can then be repeated within each pair of stages. Note that in this case there is transmission of error from our level of decomposition to all subsequent levels.

Finally by the dual equivalence of Section 2.4.3, the above discussion applies to the Symmetric Benders algorithms, which is used on structures of the form 3.1b).

In effect in symmetric decomposition, we are exchanging our LP problem for a block/dual block angular structure of special form of Figure 3.3 with dense blocks and smaller subproblems. The latter do not lose any structures of their own and are amenable to solution by specialized techniques. Symmetric decomposition may have advantages for solving certain sorts of staircase models and for combining different models into a single integrated LP model.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We briefly go over the well known dual relationship between Dantzig-Wolfe Decomposition and Benders Decomposition, in order to develop suitable notation and then elaborate upon the dual relationship between nested versions of Dantzig-Wolfe and Benders Decomposition. Next we develop a new pair of dually related decompositions termed symmetric Dantzig-Wolfe and symmetric Benders Decomposition. Finally we discuss the advantages and disadvantages of applying nested and symmetric decompositions to structured LP problems, in particular to staircase structures.

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